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2005 J. Phys. A: Math. Gen. 38 9039

(http://iopscience.iop.org/0305-4470/38/41/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 9039–9050 doi:10.1088/0305-4470/38/41/014

Construction of $3 \otimes 3$ entangled edge states with positive partial transposes

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Received 25 March 2005, in final form 25 July 2005
Published 28 September 2005
Online at stacks.iop.org/JPhysA/38/9039

Abstract
We construct a class of $3 \otimes 3$ entangled edge states with positive partial transposes using indecomposable positive linear maps. This class contains several new types of entangled edge states with respect to the range dimensions of themselves and their partial transposes.

PACS numbers: 03.67.Mn, 03.67.Ud, 02.10.Yn
Mathematics Subject Classification: 81P15, 15A30, 46L05

1. Introduction

The notion of entanglement in quantum physics has been studied extensively during the last decade in connection with the quantum information theory and quantum communication theory. A density matrix A in $(M_n \otimes M_m)^+$ is said to be *entangled* if it does not belong to $M_n^+ \otimes M_m^+$, where M_n^+ denotes the cone of all positive semi-definite $n \times n$ matrices over the complex fields. A density matrix is said to be *separable* if it belongs to $M_n^+ \otimes M_m^+$. Recall that a density matrix defines a state on the matrix algebra by the Schur or Hadamard product.

The basic question is, of course, how to distinguish entangled ones among density matrices, or equivalently among states on matrices. For a block matrix $A \in M_n \otimes M_m$, the *partial transpose* or *block transpose* A^τ of A is defined by

$$\left(\sum_{i,j=1}^{m} a_{ij} \otimes e_{ij}\right)^{\tau} = \sum_{i,j=1}^{m} a_{ji} \otimes e_{ij}.$$

In the early 1980s, it was observed by Choi [7] that the partial transpose of every separable state is positive semi-definite. This necessary condition for separability has also been found independently by Peres [27], and is now called the PPT criterion for separability. Choi [7] also gave an example of $3 \otimes 3$ entangled state whose partial transpose is positive semi-definite. This kind of entangled state is called PPTES.

A positive linear map between matrix algebras is said to be *decomposable* if it is the sum of a completely positive linear map and a completely copositive linear map. Choi [6] was the first who gave an example of an indecomposable positive linear map. Woronowicz [34] showed that every positive linear map from M_2 into M_n is decomposable if and only if $n \leqslant 3$. He showed that there is an indecomposable positive linear map from M_2 into M_4 by exhibiting an example of $2 \otimes 4$ PPTES. Størmer [30] also gave a necessary and sufficient condition for decomposability in terms of partial transpose, and gave an example of $3 \otimes 3$ PPTES. During the 1990s, several examples of PPTES have been found; see [1, 2, 8, 14–16, 29]. Among examples of PPTES, the so-called *edge* PPTES play special roles as was studied in [25].

The cone of all positive semi-definite block matrices with positive partial transposes will be denoted by $\mathbb{T}$ in this paper. The facial structures may be explained in terms of duality between the space of linear maps and the space of block matrices, as was studied in [9] which was motivated by the works of Woronowicz [34], Størmer [30] and Itoh [18]. The cone generated by separable states will be denoted by $\mathbb{V}_1$. Then the above-mentioned examples will lie in $\mathbb{T}\backslash\mathbb{V}_1$. A PPTES A in $\mathbb{T}\backslash\mathbb{V}_1$ is an *edge* PPTES if and only if the proper face of $\mathbb{T}$ containing A as an interior point does not contain a separable state.

Edge states may be classified by their range dimensions as was studied in [29]. An edge PPTES A is said to be an (s, t) edge state if the range dimension of A is s, and the range dimension of A^τ is t. Some necessary conditions for possible combination of (s, t) have been discussed in [17, 29]. In the $3 \otimes 3$ cases, it is quite curious that all known examples of edge PPTES are $(4, 4)$ or $(7, 6)$ edge states. Here, we assume that $s \geqslant t$ by the symmetry. The purpose of this paper is to construct other kinds of $3 \otimes 3$ edge states. More precisely, we construct $(7, 5)$, $(6, 5)$ and $(8, 5)$ edge states as well as $(7, 6)$ and $(4, 4)$ edge states. It seems to be still open if there exists a $(6, 6)$ or $(5, 5)$ edge state. This paper was motivated by the paper [29], where it was conjectured that every $3 \otimes 3$ entangled state has Schmidt number 2. This is equivalent to asking if every 2-positive linear map between M_3 is decomposable, by the duality mentioned above; see [4], corollary 4.3 and [10] in this direction.

The basic tool is the duality mentioned above. In section 2, we briefly recall the basic notions of the duality, together with the results in [13, 14] which show that every edge state may be constructed from an indecomposable positive linear map. Our examples of edge states will be constructed in section 3 from the indecomposable maps considered in [4].

Throughout this paper, we will not use bra–ket notation. Every vector will be considered as a column vector. If $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$ then x will be considered as an $m \times 1$ matrix, and y^* will be considered as a $1 \times n$ matrix, and so xy^* is an $m \times n$ rank 1 matrix whose range is generated by x and whose kernel is orthogonal to y. In the case of a vector x, the notation $\overline{x}$ will be used for the vector whose entries are conjugate to the corresponding entries. The notation $\langle \cdot, \cdot \rangle$ will be used for bi-linear pairing. On the other hand, $(\cdot|\cdot)$ will be used for the inner product, which is sesquilinear, that is, linear in the first variable and conjugate-linear in the second variable. For natural numbers m and n, we denote by $m \vee n$ and $m \wedge n$ the maximum and minimum of m and n, respectively.

2. Decomposable maps and PPT entanglement

For a given finite set $\mathcal{V} = \{V_1, V_2, \ldots, V_\nu\} \subset M_{m\times n}$ of $m \times n$ matrices, we define linear maps $\phi_{\mathcal{V}}$ and $\phi^{\mathcal{V}}$ from M_m into M_n by the following,

$$\phi_{\mathcal{V}} : X \mapsto \sum_{i=1}^{\nu} V_i^* X V_i, \qquad X \in M_m,$$

$$\phi^{\mathcal{V}} : X \mapsto \sum_{i=1}^{\nu} V_i^* X^{\mathrm{t}} V_i, \qquad X \in M_m,$$

where X^{t} denotes the transpose of X. We denote $\phi_V = \phi_{\{V\}}$ and $\phi^V = \phi^{\{V\}}$. It is well known [5, 24] that every completely positive (respectively completely copositive) linear map between matrix algebras is of the form $\Phi_{\mathcal{V}}$ (respectively $\Phi^{\mathcal{V}}$). We denote by $\mathbb{P}_{m\wedge n}$ (respectively $\mathbb{P}^{m\wedge n}$) the convex cone of all completely positive (respectively completely copositive) linear maps. For a subspace E of $M_{m\times n}$, we define

$$\Phi_E = \{\phi_{\mathcal{V}} \in \mathbb{P}_{m\wedge n} : \mathrm{span}\,\mathcal{V} \subset E\}, \qquad \Phi^E = \{\phi^{\mathcal{V}} \in \mathbb{P}^{m\wedge n} : \mathrm{span}\,\mathcal{V} \subset E\},$$

where span $\mathcal{V}$ denotes the span of the set $\mathcal{V}$. We have shown in [21] that the correspondence

$$E \mapsto \Phi_E \qquad (\text{respectively } E \mapsto \Phi^E)$$

gives rise to a lattice isomorphism between the lattice of all subspaces of the vector space $M_{m\times n}$ and the lattice of all faces of the convex cones $\mathbb{P}_{m\wedge n}$ (respectively $\mathbb{P}^{m\wedge n}$). A linear map in the cone

$$\mathbb{D} := \mathrm{conv}(\mathbb{P}_{m\wedge n}, \mathbb{P}^{m\wedge n})$$

is said to be *decomposable*, where $\mathrm{conv}(C_1, C_2)$ denotes the convex hull of C_1 and C_2. Every decomposable map is positive, that is, sends positive semi-definite matrices into themselves, but the converse is not true. There are many examples of indecomposable positive linear maps in the literature [4, 6, 10–12, 19, 20, 26, 28, 30–33]. We have shown in [23] that every face of the cone $\mathbb{D}$ is of the form

$$\sigma(D, E) := \mathrm{conv}(\Phi_D, \Phi^E)$$

for a pair (D, E) of subspaces of $M_{m\times n}$. This pair of subspaces is uniquely determined under the assumption

$$\sigma(D, E) \cap \mathbb{P}_{m\wedge n} = \Phi_D, \qquad \sigma(D, E) \cap \mathbb{P}^{m\wedge n} = \Phi^E. \tag{1}$$

We say that a pair (D, E) is a *decomposition pair* if the set $\mathrm{conv}(\Phi_D, \Phi^E)$ is a face of $\mathbb{D}$ with the condition (1). Faces of the cone $\mathbb{D}$ and decomposition pairs correspond to each other in this way. Whenever we use the notation $\sigma(D, E)$, we assume that (D, E) is a decomposition pair. It is very hard to determine all decomposition pairs; see [3, 22] for the simplest case of $m = n = 2$.

Now, we turn our attention to the block matrices, and identify an $m \times n$ matrix $z \in M_{m\times n}$ and a vector $\tilde{z} \in \mathbb{C}^n \otimes \mathbb{C}^m$ as follows. For $z = [z_{ik}] \in M_{m\times n}$, define

$$z_i = \sum_{k=1}^{n} z_{ik} e_k \in \mathbb{C}^n, \qquad i = 1, 2, \ldots, m,$$

$$\tilde{z} = \sum_{i=1}^{m} z_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m.$$

Then $z \mapsto \tilde{z}$ defines an inner product isomorphism from $M_{m\times n}$ onto $\mathbb{C}^n \otimes \mathbb{C}^m$. We also note that $\tilde{z}\tilde{z}^*$ is a positive semi-definite matrix in $M_n \otimes M_m$ of rank 1. We consider the convex cones

$$\mathbb{V}_s = \mathrm{conv}\{\tilde{z}\tilde{z}^* : \mathrm{rank}\, z \leqslant s\}, \qquad \mathbb{V}^s = \mathrm{conv}\{(\tilde{z}\tilde{z}^*)^\tau : \mathrm{rank}\, z \leqslant s\}.$$

for $s = 1, 2, \ldots, m \wedge n$. By the relation

$$\widetilde{xy^*}\widetilde{xy^*}^* = (\bar{y} \otimes x)(\bar{y} \otimes x)^* = \overline{yy^*} \otimes xx^*, \qquad x \in \mathbb{C}^m, \quad y \in \mathbb{C}^n,$$

we have

$$\mathbb{V}_1 = M_n^+ \otimes M_m^+.$$

Therefore, a density matrix in $M_n \otimes M_m$ is separable if and only if it belongs to the cone $\mathbb{V}_1$. If $z = xy^*$ is a rank 1 matrix with column vectors $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$ then $(zz^*)^\tau = ww^*$ is positive semi-definite with $w = \bar{x}y^*$ by a direct simple calculation. Therefore, we see [7, 27] that every separable state belongs to the convex cone

$$\mathbb{T} := \mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n} = \{A \in (M_n \otimes M_m)^+ : A^\tau \in (M_n \otimes M_m)^+\}.$$

A block matrix in the cone $\mathbb{T}$ is said to be *of positive partial transpose*.

It is well known that every face of $\mathbb{V}_{m\wedge n} = (M_n \otimes M_m)^+$ and $\mathbb{V}^{m\wedge n}$ is of the form

$$\Psi_D = \{A \in (M_n \otimes M_m)^+ : \mathcal{R}A \subset \widetilde{D}\},$$
$$\Psi^E = \{A \in M_n \otimes M_m : A^\tau \in \Psi_E\},$$

respectively, where $\mathcal{R}A$ is the range space of A and $\widetilde{D} = \{\tilde{z} \in \mathbb{C}^n \otimes \mathbb{C}^m : z \in D\}$. It is also easy to see that every face of $\mathbb{T}$ is of the form

$$\tau(D, E) := \Psi_D \cap \Psi^E$$

for a pair (D, E) of subspaces of $M_{m\times n}$, as was explained in [13]. This pair is uniquely determined under the assumption

$$\operatorname{int} \tau(D, E) \subset \operatorname{int} \Psi_D, \qquad \operatorname{int} \tau(D, E) \subset \operatorname{int} \Psi^E, \tag{2}$$

where $\operatorname{int} C$ denote the relative interior of the convex set C with respect to the hyperplane generated by C. We say that a pair (D, E) of subspaces is an *intersection pair* if it satisfies the assumption (2), and $\tau(D, E) \neq \emptyset$. We also assume condition (2) whenever we use the notation $\tau(D, E)$.

Note that the convex cones $\mathbb{D}$ and $\mathbb{T}$ are sitting in the vector space $\mathcal{L}(M_m, M_n)$ of all linear maps from M_m into M_n and the vector space $M_n \otimes M_m$ of all block matrices. In [9], we have considered the bi-linear pairing between the spaces $\mathcal{L}(M_m, M_n)$ and $M_n \otimes M_m$, given by

$$\langle A, \phi \rangle = \operatorname{Tr}\left[\left(\sum_{i,j=1}^{m} \phi(e_{ij}) \otimes e_{ij}\right) A^{\mathrm{t}}\right] = \sum_{i,j=1}^{m} \langle \phi(e_{ij}), a_{ij} \rangle, \tag{3}$$

for $A = \sum_{i,j=1}^m a_{ij} \otimes e_{ij} \in M_n \otimes M_m$ and $\phi \in \mathcal{L}(M_m, M_n)$, where the bi-linear form on the right-hand side is given by $\langle X, Y \rangle = \operatorname{Tr}(YX^{\mathrm{t}})$ for $X, Y \in M_n$. The main result of [9] tells us that two cones $\mathbb{D}$ and $\mathbb{T}$ are dual to each other in the following sense:

$$\phi \in \mathbb{D} \iff \langle A, \phi \rangle \geqslant 0 \quad \text{for every } A \in \mathbb{T},$$
$$A \in \mathbb{T} \iff \langle A, \phi \rangle \geqslant 0 \quad \text{for every } \phi \in \mathbb{D}.$$

It was also shown in [9] that the cone $\mathbb{P}_s$ (respectively $\mathbb{P}^s$) consisting of s-positive (respectively s-copositive) linear maps is dual to the cone $\mathbb{V}_s$ (respectively $\mathbb{V}^s$) in the above sense; see also [29]. Some faces of the cone $\mathbb{D}$ arise from this duality which is of the form

$$\tau(D, E)' := \{\phi \in \mathbb{D} : \langle A, \phi \rangle = 0 \text{ for every } A \in \tau(D, E)\}$$

for a face $\tau(D, E)$ of $\mathbb{T}$. If A is an interior point of $\tau(D, E)$ then we have

$$\tau(D, E)' = A' := \{\phi \in \mathbb{D} : \langle A, \phi \rangle = 0\}.$$

It is easy to see that

$$\tau(D, E)' = \sigma(D^\perp, E^\perp).$$

It should be noted that not every face arises in this way even in the simplest case of $m = n = 2$; see [3, 22]. Nevertheless, every face of the cone $\mathbb{T}$ arises from this duality. More precisely, it was shown in [13] that every face of the cone $\mathbb{T}$ is of the form

$$\sigma(D, E)' := \{A \in \mathbb{T} : \langle A, \phi\rangle = 0 \text{ for every } \phi \in \sigma(D, E)\} = \tau(D^\perp, E^\perp)$$

for a face $\sigma(D, E)$ of the cone $\mathbb{D}$. The following is implicit in [13]. We state here for the clearance

Proposition 2.1. *A pair (D, E) of subspaces of $M_{m\times n}$ is an intersection pair if and only if there exists $A \in \mathbb{T}$ such that $\mathcal{R}A = \widetilde{D}$ and $\mathcal{R}A^\tau = \widetilde{E}$. If this is the case then we have*

$$\operatorname{int}\tau(D, E) = \{A \in \mathbb{T} : \mathcal{R}A = \widetilde{D}, \mathcal{R}A^\tau = \widetilde{E}\}.$$

Proof. Let (D, E) be an intersection pair and take $A \in \operatorname{int}\tau(D, E)$. Then $A' = \tau(D, E)' = \sigma(D^\perp, E^\perp)$, and we have $\mathcal{R}A = \widetilde{D}$ and $\mathcal{R}A^\tau = \widetilde{E}$ by [13] lemma 1. For the converse, assume that there is $A \in \mathbb{T}$ such that $\mathcal{R}A = \widetilde{D}$ and $\mathcal{R}A^\tau = \widetilde{E}$. Take the intersection pair (D_1, E_1) such that $A \in \operatorname{int}\tau(D_1, E_1)$ Then we have $\mathcal{R}A = \widetilde{D}_1$ and $\mathcal{R}A^\tau = \widetilde{E}_1$, and so $D = D_1$ and $E = E_1$. The last statement has been already proved. □

Corollary 2.2. *If (D_1, E_1) and (D_2, E_2) are intersection pairs then $(D_1 \vee D_2, E_1 \vee E_2)$ is also an intersection pair.*

Proof. Take $A_i \in \mathbb{T}$ with $\mathcal{R}A_i = \widetilde{D}_i$ and $\mathcal{R}A_i^\tau = \widetilde{E}_i$ for $i = 1, 2$. Then we have

$$A_1 + A_2 \in \mathbb{T}, \qquad \mathcal{R}(A_1 + A_2) = \widetilde{D_1 \vee D_2}, \qquad \mathcal{R}(A_1 + A_2)^\tau = \widetilde{E_1 \vee E_2}.$$

Therefore, we see that $(D_1 \vee D_2, E_1 \vee E_2)$ is an intersection pair. □

Now, we have two cones $\mathbb{D} \subset \mathbb{P}_1$ in the space $\mathcal{L}(M_m, M_n)$ and other two cones $\mathbb{V}_1 \subset \mathbb{T}$ in the space $M_n \otimes M_m$. Recall that $\mathbb{P}_1$ denotes the cone of all positive linear maps. The pairs $(\mathbb{D}, \mathbb{T})$ and $(\mathbb{P}_1, \mathbb{V}_1)$ are dual to each other, as was explained before. Let $\sigma(D, E)$ be a proper face of the cone $\mathbb{D}$. Then we have the following two cases,

$$\operatorname{int}\sigma(D, E) \subset \operatorname{int}\mathbb{P}_1 \qquad \text{or} \qquad \sigma(D, E) \subset \partial\mathbb{P}_1,$$

since $\sigma(D, E)$ is a convex subset of the cone $\mathbb{P}_1$, where $\partial C := C\backslash \operatorname{int} C$ denotes the boundary of the convex set C. We have shown in [13], [14] that

$$\operatorname{int}\sigma(D, E) \subset \operatorname{int}\mathbb{P}_1 \iff \sigma(D, E)' \cap \mathbb{V}_1 = \{0\}. \tag{4}$$

Every element $A \in \mathbb{T}$ determines a unique face $\tau(D, E)$ whose interior contains A. Then a density block matrix $A \in \mathbb{T}$ is an (entangled) edge state if and only if $\tau(D, E) \cap \mathbb{V}_1 = \{0\}$. Therefore, we conclude the following:

(i) If $\sigma(D, E)$ is a face of $\mathbb{D}$ with $\operatorname{int}\sigma(D, E) \subset \operatorname{int}\mathbb{P}_1$ then every nonzero element in the dual face $\sigma(D, E)'$ gives rise to an entangled edge state up to constant multiplications.

(ii) Every edge state arises in this way.

The second claim follows from the fact that every face of the cone $\mathbb{T}$ arises from the duality, as was explained before.

3. Construction of $3 \otimes 3$ PPT entangled edge states

We begin with the decomposable positive linear map $\phi : M_3 \to M_3$ defined by

$$\phi = \phi_{e_{11}-e_{22}} + \phi_{e_{22}-e_{33}} + \phi_{e_{33}-e_{11}} + \phi^{\mu e_{12}-\lambda e_{21}} + \phi^{\mu e_{23}-\lambda e_{32}} + \phi^{\mu e_{31}-\lambda e_{13}},$$

which lies in $\partial\mathbb{D} \cap \operatorname{int} \mathbb{P}_1$ as was shown in [14], where

$$\lambda\mu = 1, \qquad \lambda > 0, \qquad \lambda \neq 1.$$

We try to determine the dual face $\tau(D, E) = \{\phi\}'$. This map was originated from indecomposable positive linear maps considered in [4]. We note that D is the seven-dimensional space given by

$$D = \operatorname{span}\{e_{12}, e_{21}, e_{23}, e_{32}, e_{31}, e_{13}, e_{11} + e_{22} + e_{33}\},$$

and E is the six-dimensional space given by

$$E = \operatorname{span}\{\lambda e_{12} + \mu e_{21}, \lambda e_{23} + \mu e_{32}, \lambda e_{31} + \mu e_{13}, e_{11}, e_{22}, e_{33}\}.$$

Therefore, every matrix $x_i \in E$ is of the form

$$x_i = \rho \circ \sigma_i$$

where

$$\rho = \begin{pmatrix} 1 & \lambda & \mu \\ \mu & 1 & \lambda \\ \lambda & \mu & 1 \end{pmatrix}, \qquad \sigma_i = \begin{pmatrix} \xi_i & \alpha_i & \gamma_i \\ \alpha_i & \eta_i & \beta_i \\ \gamma_i & \beta_i & \zeta_i \end{pmatrix},$$

and $\rho \circ \sigma_i$ denotes the Hadamard product of ρ and σ_i.

It follows that if $X^\tau = \sum_i \widetilde{x}_i \widetilde{x}_i^* \in \mathbb{V}_3 \cap \mathbb{V}^3$ belongs to $\tau(D, E)$ then

$$X^\tau = \sum (\widetilde{\rho}\widetilde{\rho}^*) \circ (\widetilde{\sigma}_i \widetilde{\sigma}_i^*) = (\widetilde{\rho}\widetilde{\rho}^*) \circ Y$$

with

$$\widetilde{\rho}\widetilde{\rho}^* = \left(\begin{array}{ccc|ccc|ccc} 1 & \lambda & \mu & \mu & 1 & \lambda & \lambda & \mu & 1 \\ \lambda & \lambda^2 & 1 & 1 & \lambda & \lambda^2 & \lambda^2 & 1 & \lambda \\ \mu & 1 & \mu^2 & \mu^2 & \mu & 1 & 1 & \mu^2 & \mu \\ \hline \mu & 1 & \mu^2 & \mu^2 & \mu & 1 & 1 & \mu^2 & \mu \\ 1 & \lambda & \mu & \mu & 1 & \lambda & \lambda & \mu & 1 \\ \lambda & \lambda^2 & 1 & 1 & \lambda & \lambda^2 & \lambda^2 & 1 & \lambda \\ \hline \lambda & \lambda^2 & 1 & 1 & \lambda & \lambda^2 & \lambda^2 & 1 & \lambda \\ \mu & 1 & \mu^2 & \mu^2 & \mu & 1 & 1 & \mu^2 & \mu \\ 1 & \lambda & \mu & \mu & 1 & \lambda & \lambda & \mu & 1 \end{array}\right),$$

and $Y = \sum \widetilde{\sigma}_i \widetilde{\sigma}_i^*$ is given by

$$\left(\begin{array}{ccc:ccc:ccc}
(\xi|\xi) & (\xi|\alpha) & (\xi|\gamma) & (\xi|\alpha) & (\xi|\eta) & (\xi|\beta) & (\xi|\gamma) & (\xi|\beta) & (\xi|\zeta) \\
(\alpha|\xi) & (\alpha|\alpha) & (\alpha|\gamma) & (\alpha|\alpha) & (\alpha|\eta) & (\alpha|\beta) & (\alpha|\gamma) & (\alpha|\beta) & (\alpha|\zeta) \\
(\gamma|\xi) & (\gamma|\alpha) & (\gamma|\gamma) & (\gamma|\alpha) & (\gamma|\eta) & (\gamma|\beta) & (\gamma|\gamma) & (\gamma|\beta) & (\gamma|\zeta) \\
\\
(\alpha|\xi) & (\alpha|\alpha) & (\alpha|\gamma) & (\alpha|\alpha) & (\alpha|\eta) & (\alpha|\beta) & (\alpha|\gamma) & (\alpha|\beta) & (\alpha|\zeta) \\
(\eta|\xi) & (\eta|\alpha) & (\eta|\gamma) & (\eta|\alpha) & (\eta|\eta) & (\eta|\beta) & (\eta|\gamma) & (\eta|\beta) & (\eta|\zeta) \\
(\beta|\xi) & (\beta|\alpha) & (\beta|\gamma) & (\beta|\alpha) & (\beta|\eta) & (\beta|\beta) & (\beta|\gamma) & (\beta|\beta) & (\beta|\zeta) \\
\\
(\gamma|\xi) & (\gamma|\alpha) & (\gamma|\gamma) & (\gamma|\alpha) & (\gamma|\eta) & (\gamma|\beta) & (\gamma|\gamma) & (\gamma|\beta) & (\gamma|\zeta) \\
(\beta|\xi) & (\beta|\alpha) & (\beta|\gamma) & (\beta|\alpha) & (\beta|\eta) & (\beta|\beta) & (\beta|\gamma) & (\beta|\beta) & (\beta|\zeta) \\
(\zeta|\xi) & (\zeta|\alpha) & (\zeta|\gamma) & (\zeta|\alpha) & (\zeta|\eta) & (\zeta|\beta) & (\zeta|\gamma) & (\zeta|\beta) & (\zeta|\zeta)
\end{array}\right)$$

if we denote by $\xi, \eta, \zeta, \alpha, \beta$ and γ the vectors whose entries are $\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i$ and γ_i, respectively. So, $X = (X^\tau)^\tau$ is of the form

$$\left(\begin{array}{ccccccccc}
(\xi|\xi) & \lambda(\xi|\alpha) & \mu(\xi|\gamma) & \mu(\alpha|\xi) & (\alpha|\alpha) & \mu^2(\alpha|\gamma) & \lambda(\gamma|\xi) & \lambda^2(\gamma|\alpha) & (\gamma|\gamma) \\
\lambda(\alpha|\xi) & \lambda^2(\alpha|\alpha) & (\alpha|\gamma) & (\eta|\xi) & \lambda(\eta|\alpha) & \mu(\eta|\gamma) & \mu(\beta|\xi) & (\beta|\alpha) & \mu^2(\beta|\gamma) \\
\mu(\gamma|\xi) & (\gamma|\alpha) & \mu^2(\gamma|\gamma) & \lambda(\beta|\xi) & \lambda^2(\beta|\alpha) & (\beta|\gamma) & (\zeta|\xi) & \lambda(\zeta|\alpha) & \mu(\zeta|\gamma) \\
\\
\mu(\xi|\alpha) & (\xi|\eta) & \lambda(\xi|\beta) & \mu^2(\alpha|\alpha) & \mu(\alpha|\eta) & (\alpha|\beta) & (\gamma|\alpha) & \lambda(\gamma|\eta) & \lambda^2(\gamma|\beta) \\
(\alpha|\alpha) & \lambda(\alpha|\eta) & \lambda^2(\alpha|\beta) & \mu(\eta|\alpha) & (\eta|\eta) & \lambda(\eta|\beta) & \mu^2(\beta|\alpha) & \mu(\beta|\eta) & (\beta|\beta) \\
\mu^2(\gamma|\alpha) & \mu(\gamma|\eta) & (\gamma|\beta) & (\beta|\alpha) & \lambda(\beta|\eta) & \lambda^2(\beta|\beta) & \mu(\zeta|\alpha) & (\zeta|\eta) & \lambda(\zeta|\beta) \\
\\
\lambda(\xi|\gamma) & \mu(\xi|\beta) & (\xi|\zeta) & (\alpha|\gamma) & \mu^2(\alpha|\beta) & \mu(\alpha|\zeta) & \lambda^2(\gamma|\gamma) & (\gamma|\beta) & \lambda(\gamma|\zeta) \\
\lambda^2(\alpha|\gamma) & (\alpha|\beta) & \lambda(\alpha|\zeta) & \lambda(\eta|\gamma) & \mu(\eta|\beta) & (\eta|\zeta) & (\beta|\gamma) & \mu^2(\beta|\beta) & \mu(\beta|\zeta) \\
(\gamma|\gamma) & \mu^2(\gamma|\beta) & \mu(\gamma|\zeta) & \lambda^2(\beta|\gamma) & (\beta|\beta) & \lambda(\beta|\zeta) & \lambda(\zeta|\gamma) & \mu(\zeta|\beta) & (\zeta|\zeta)
\end{array}\right)$$

Now, we consider the condition $X \in (\Phi_{D^\perp})'$ to see that

$$\langle Y, \widetilde{\rho}\widetilde{\rho}^* \circ \phi^V \rangle = \langle \widetilde{\rho}\widetilde{\rho}^* \circ Y, \phi^V \rangle = \langle X^\tau, \phi^V \rangle = \langle X, \phi_V \rangle = 0$$

for any $V \in D^\perp$. Note that any matrix V in $D^\perp$ is of the form

$$V = \begin{pmatrix} a_1 & \cdot & \cdot \\ \cdot & a_2 & \cdot \\ \cdot & \cdot & a_3 \end{pmatrix}, \qquad a_1 + a_2 + a_3 = 0,$$

and

$$\widetilde{\rho}\widetilde{\rho}^* \circ \phi^V = \left(\begin{array}{ccccccccc}
|a_1|^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \overline{a}_2 a_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{a}_3 a_1 & \cdot & \cdot \\
\\
\cdot & \overline{a}_1 a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & |a_2|^2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{a}_3 a_2 & \cdot \\
\\
\cdot & \cdot & \overline{a}_1 a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \overline{a}_2 a_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & |a_3|^2
\end{array}\right).$$

Therefore, it follows that

$$\begin{pmatrix} (\xi|\xi) & (\alpha|\alpha) & (\gamma|\gamma) \\ (\alpha|\alpha) & (\eta|\eta) & (\beta|\beta) \\ (\gamma|\gamma) & (\beta|\beta) & (\zeta|\zeta) \end{pmatrix} \circ \begin{pmatrix} |a_1|^2 & \overline{a}_2 a_1 & \overline{a}_3 a_1 \\ \overline{a}_1 a_2 & |a_2|^2 & \overline{a}_3 a_2 \\ \overline{a}_1 a_3 & \overline{a}_2 a_3 & |a_3|^2 \end{pmatrix} = 0$$

whenever $a_1 + a_2 + a_3 = 0$, where $A \circ B = \sum a_{ij} b_{ij}$ by an abuse of notation. Taking $(a_1, a_2, a_3) = (1, -1, 0)$, we have $\|\xi\|^2 + \|\eta\|^2 = 2\|\alpha\|^2$. But the positivity of the 2×2 submatrix of X with the 1, 5 columns and rows tells us that $\|\xi\| = \|\eta\| = \|\alpha\|$. Similarly, we have

$$\|\xi\| = \|\eta\| = \|\zeta\| = \|\alpha\| = \|\beta\| = \|\gamma\| = 1$$

by assuming that $\|\xi\| = 1$. Hence, X is of the form

$$\begin{pmatrix}
1 & \lambda(\xi|\alpha) & \mu(\xi|\gamma) & \mu(\alpha|\xi) & 1 & \mu^2(\alpha|\gamma) & \lambda(\gamma|\xi) & \lambda^2(\gamma|\alpha) & 1 \\
\lambda(\alpha|\xi) & \lambda^2 & (\alpha|\gamma) & (\eta|\xi) & \lambda(\eta|\alpha) & \mu(\eta|\gamma) & \mu(\beta|\xi) & (\beta|\alpha) & \mu^2(\beta|\gamma) \\
\mu(\gamma|\xi) & (\gamma|\alpha) & \mu^2 & \lambda(\beta|\xi) & \lambda^2(\beta|\alpha) & (\beta|\gamma) & (\zeta|\xi) & \lambda(\zeta|\alpha) & \mu(\zeta|\gamma) \\
\mu(\xi|\alpha) & (\xi|\eta) & \lambda(\xi|\beta) & \mu^2 & \mu(\alpha|\eta) & (\alpha|\beta) & (\gamma|\alpha) & \lambda(\gamma|\eta) & \lambda^2(\gamma|\beta) \\
1 & \lambda(\alpha|\eta) & \lambda^2(\alpha|\beta) & \mu(\eta|\alpha) & 1 & \lambda(\eta|\beta) & \mu^2(\beta|\alpha) & \mu(\beta|\eta) & 1 \\
\mu^2(\gamma|\alpha) & \mu(\gamma|\eta) & (\gamma|\beta) & (\beta|\alpha) & \lambda(\beta|\eta) & \lambda^2 & \mu(\zeta|\alpha) & (\zeta|\eta) & \lambda(\zeta|\beta) \\
\lambda(\xi|\gamma) & \mu(\xi|\beta) & (\xi|\zeta) & (\alpha|\gamma) & \mu^2(\alpha|\beta) & \mu(\alpha|\zeta) & \lambda^2 & (\gamma|\beta) & \lambda(\gamma|\zeta) \\
\lambda^2(\alpha|\gamma) & (\alpha|\beta) & \lambda(\alpha|\zeta) & \lambda(\eta|\gamma) & \mu(\eta|\beta) & (\eta|\zeta) & (\beta|\gamma) & \mu^2 & \mu(\beta|\zeta) \\
1 & \mu^2(\gamma|\beta) & \mu(\gamma|\zeta) & \lambda^2(\beta|\gamma) & 1 & \lambda(\beta|\zeta) & \lambda(\zeta|\gamma) & \mu(\zeta|\beta) & 1
\end{pmatrix}.$$

If we take vectors so that span $\{\xi, \eta, \zeta\} \perp$ span $\{\alpha, \beta, \gamma\}$ with mutually orthonormal vectors α, β, γ then we have

$$X = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \lambda^2 & \cdot & (\eta|\xi) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \mu^2 & \cdot & \cdot & \cdot & (\zeta|\xi) & \cdot & \cdot \\
\cdot & (\xi|\eta) & \cdot & \mu^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \lambda^2 & \cdot & (\zeta|\eta) & \cdot \\
\cdot & \cdot & (\xi|\zeta) & \cdot & \cdot & \cdot & \lambda^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & (\eta|\zeta) & \cdot & \mu^2 & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1
\end{pmatrix} \tag{5}$$

and

$$X^\tau = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & (\xi|\eta) & \cdot & \cdot & \cdot & (\xi|\zeta) \\ \cdot & \lambda^2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu^2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \\ \cdot & 1 & \cdot & \mu^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ (\eta|\xi) & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & (\eta|\zeta) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda^2 & \cdot & 1 & \cdot \\ \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \lambda^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \mu^2 & \cdot \\ (\zeta|\xi) & \cdot & \cdot & \cdot & (\zeta|\eta) & \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

We note that the rank of X is equal to

$$1 + \text{rank}\begin{pmatrix} (\xi|\xi) & (\xi|\eta) \\ (\eta|\xi) & (\eta|\eta) \end{pmatrix} + \text{rank}\begin{pmatrix} (\eta|\eta) & (\eta|\zeta) \\ (\zeta|\eta) & (\zeta|\zeta) \end{pmatrix} + \text{rank}\begin{pmatrix} (\zeta|\zeta) & (\zeta|\xi) \\ (\xi|\zeta) & (\xi|\xi) \end{pmatrix}$$

and the rank of X^τ is equal to

$$3 + \text{rank}\begin{pmatrix} (\xi|\xi) & (\xi|\eta) & (\xi|\zeta) \\ (\eta|\xi) & (\eta|\eta) & (\eta|\zeta) \\ (\zeta|\xi) & (\zeta|\eta) & (\zeta|\zeta) \end{pmatrix}.$$

Recall that the rank of the $n \times n$ matrix $[(\xi_i|\xi_j)]_{i,j=1}^n$ is the dimension of the space span $\{\xi_1, \ldots, \xi_n\}$. If we take mutually independent vectors ξ, η, ζ then we get a (7, 6) edge state. If we take vectors so that dim span $\{\xi, \eta, \zeta\} = 2$ and neither of the two vectors is linearly dependent then we may get a (7, 5) edge state. If we take vectors so that dim span $\{\xi, \eta, \zeta\} = 2$ and one pair of two vectors are linearly dependent then we have a (6, 5) edge state. Finally, if we take vectors with $\xi = \eta = \zeta$ then we have a (4, 4) edge state as was given in the paper [14]. For more explicit examples, we put

$$\xi = e_1, \qquad \eta = e_2$$

in $\mathbb{C}^3$. We get 1-parameter family of (7, 6) edge states (respectively (7, 5) and (6, 5) edge states) if we put

$$\zeta = e_3 \qquad (\text{respectively} \quad \zeta = \tfrac{1}{\sqrt{2}}(e_1 + e_2) \quad \text{and} \quad \zeta = e_1)$$

in the matrix (5).

In order to get other edge states such as (8, 5), we discard the condition $X \in (\Phi_{D^\perp})'$. We define vectors $\xi, \eta, \zeta \in \mathbb{C}^5$ by

$$\xi = \sqrt{t}e_1, \qquad \eta = \sqrt{t}e_2, \qquad \zeta = \sqrt{\frac{1}{t(t+1)}}(\xi + \eta)$$

for $t > 1$. We also take mutually orthonormal vectors $\alpha, \beta, \gamma \in \mathbb{C}^5$ in (5) so that

$$\operatorname{span}\{\xi, \eta, \zeta\} \perp \operatorname{span}\{\alpha, \beta, \gamma\}.$$

Then we have

$$X = \begin{pmatrix} t & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \lambda^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu^2 & \cdot & \cdot & \cdot & \sqrt{\frac{t}{t+1}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mu^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda^2 & \cdot & \sqrt{\frac{t}{t+1}} & \cdot \\ \cdot & \cdot & \sqrt{\frac{t}{t+1}} & \cdot & \cdot & \cdot & \lambda^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{t}{t+1}} & \cdot & \mu^2 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \frac{2}{t+1} \end{pmatrix}$$

and

$$X^\tau = \begin{pmatrix} t & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{t}{t+1}} \\ \cdot & \lambda^2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu^2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \mu^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \sqrt{\frac{t}{t+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda^2 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \lambda^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \mu^2 & \cdot \\ \sqrt{\frac{t}{t+1}} & \cdot & \cdot & \cdot & \sqrt{\frac{t}{t+1}} & \cdot & \cdot & \cdot & \frac{2}{t+1} \end{pmatrix}.$$

First of all, two matrices

$$\begin{pmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & \frac{2}{t+1} \end{pmatrix} \qquad \begin{pmatrix} \lambda^2 & \sqrt{\frac{t}{t+1}} \\ \sqrt{\frac{t}{t+1}} & \mu^2 \end{pmatrix}, \qquad t > 1$$

are positive semi-definite with rank 2. It follows that X belongs to $\mathbb{T}$. We note that $\mathcal{R}X$ is an eight-dimensional space spanned by

$$te_{11} + e_{22} + e_{33}, \qquad e_{11} + te_{22} + e_{33}, \qquad e_{12}, e_{21}, e_{23}, e_{32}, e_{31}, e_{13} \tag{6}$$

and $\mathcal{R}X^\tau$ is a five-dimensional space spanned by

$$te_{11} + \sqrt{\frac{t}{t+1}}e_{33}, \qquad te_{22} + \sqrt{\frac{t}{t+1}}e_{33}, \qquad \lambda e_{12} + \mu e_{21}, \qquad \lambda e_{23} + \mu e_{32}, \lambda e_{31} + \mu e_{13}.$$

Now, we proceed to show that X is an edge state. It is easy to see that $\mathcal{R}X^\tau$ has the following six rank 1 matrices,

$$\begin{pmatrix} (t^2+t)^{\frac14} & \cdot & \mu \\ \cdot & \cdot & \cdot \\ \lambda & \cdot & (t^2+t)^{-\frac14} \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & (t^2+t)^{\frac14} & \lambda \\ \cdot & \mu & (t^2+t)^{-\frac14} \end{pmatrix} \quad \begin{pmatrix} \mathrm{i} & \lambda & \cdot \\ \mu & -\mathrm{i} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\begin{pmatrix} -(t^2+t)^{\frac14} & \cdot & \mu \\ \cdot & \cdot & \cdot \\ \lambda & \cdot & -(t^2+t)^{-\frac14} \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & -(t^2+t)^{\frac14} & \lambda \\ \cdot & \mu & -(t^2+t)^{-\frac14} \end{pmatrix} \begin{pmatrix} -\mathrm{i} & \lambda & \cdot \\ \mu & \mathrm{i} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

up to scalar multiplication. We note that four matrices in the above list have real entries. If a rank 1 matrix $xy^* \in \mathcal{R}X^\tau$ is one of them then $xy^* = \bar{x}y^*$. If $xy^* \in \mathcal{R}X^\tau$ is one of the following matrices,

$$\mathrm{i}e_{11} - \mathrm{i}e_{22} + \lambda e_{12} + \mu e_{21}, \qquad -\mathrm{i}e_{11} + \mathrm{i}e_{22} + \lambda e_{12} + \mu e_{21},$$

with complex entries, then $\bar{x}y^*$ should be

$$\mathrm{i}e_{11} + \mathrm{i}e_{22} + \lambda e_{12} - \mu e_{21}, \qquad -\mathrm{i}e_{11} - \mathrm{i}e_{22} + \lambda e_{12} - \mu e_{21},$$

respectively. In both cases, we can show that $\bar{x}y^*$ does not belong to $\mathcal{R}X$ which is spanned by matrices in 6. Consequently, there is no rank 1 matrix $xy^* \in \mathcal{R}X^\tau$ with $\bar{x}y^* \in \mathcal{R}X$. This gives us a 2-parameter family of (8, 5) edge states.

Acknowledgments

This work was partially supported by KOSEF grant (R14-2003-006-01002-0).

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